

On continual classes of evolution equations

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Abstract

We reproduce our old result on the complete class of local evolution equations admitting the original Miura transformation, and then consider this continual class of evolution equations from the standpoint of zero-curvature representations.

Introduction

The aim of this e-print is twofold. First, we reproduce here, in the arXiv, our old result [1] that the original Miura transformation [2] is applicable to a continual class of local evolution equations. Second, we show how to obtain the same continual class of local evolution equations from a zero-curvature representation (ZCR) with a fixed x -part, using our method of cyclic bases of ZCRs [3, 4, 5]. This may be found useful, in view of the recent results of S. A. Igonin [6] on a relation between Miura type transformations and ZCRs.

The Miura transformation

Let us find all the local evolution equations

$$u_t = f(x, t, u, u_1, \dots, u_n) \quad (1)$$

which admit the Miura transformation

$$u = v_1 - \frac{1}{2}v^2, \quad (2)$$

where u_i and v_i ($i = 1, 2, \dots$) denote $\partial_x^i u$ and $\partial_x^i v$, respectively.

The Miura transformation (2) relates a local evolution equation (1) with a local evolution equation

$$v_t = g(x, t, v, v_1, \dots, v_n) \quad (3)$$

if the function $u(x, t)$ determined by (2) satisfies (1) whenever the function $v(x, t)$ satisfies (3). This can be equivalently expressed by the condition

$$f(x, t, u, u_1, \dots, u_n) = (D_x - v)g(x, t, v, v_1, \dots, v_n), \quad (4)$$

where D_x denotes the total derivative with respect to x . Note that the condition (4) must be a differential consequence of the relation (2), otherwise (4) and (2) would produce an ordinary differential equation restricting solutions v of the evolution equation (3).

Now, using the relation (2) and its differential consequences, i.e., $v_1 = u + \frac{1}{2}v^2$, $v_2 = u_1 + uv + \frac{1}{2}v^3$, $v_3 = u_2 + u^2 + u_1v + 2uv^2 + \frac{3}{4}v^4$, etc., we eliminate v_1, v_2, \dots, v_{n+1} from the condition (4), and in this way rewrite (4) in the following equivalent form:

$$f(x, t, u, u_1, \dots, u_n) = (\tilde{D}_x - v)h(x, t, v, u, u_1, \dots, u_{n-1}), \quad (5)$$

where

$$\tilde{D}_x = \partial_x + (u + \frac{1}{2}v^2)\partial_v + u_1\partial_u + u_2\partial_{u_1} + u_3\partial_{u_2} + \dots \quad (6)$$

and

$$h(x, t, v, u, u_1, \dots, u_{n-1}) = g(x, t, v, \tilde{D}_x v, \tilde{D}_x^2 v, \dots, \tilde{D}_x^n v). \quad (7)$$

The crucial point is that the condition (5) must be an identity, because it cannot be a differential consequence of the relation (2). Therefore, the right-hand side of (5) must be independent of v , this determines admissible functions h , and then (5) is simply a definition of admissible functions f .

Next, repeatedly applying ∂_v to (5) three times and using the evident identity

$$\partial_v \tilde{D}_x = (\tilde{D}_x + v)\partial_v, \quad (8)$$

we obtain the auxiliary condition

$$(\tilde{D}_x + 2v)\partial_v^3 h = 0. \quad (9)$$

Since h must be a local expression, it follows from (9) that $\partial_v^3 h = 0$, i.e., the function h is necessarily of the form

$$h = \frac{1}{2}v^2 p + vq + r, \quad (10)$$

where p , q and r are functions of $x, t, u, u_1, \dots, u_{n-1}$.

Finally, substituting the expression (10) into the condition (5) and taking into account that p , q , r and f do not depend on v , we obtain the following:

$$q = D_x p, \quad r = (D_x^2 + u)p, \quad f = (D_x^3 + 2uD_x + u_1)p, \quad (11)$$

where p is an arbitrary function of $x, t, u, u_1, \dots, u_{n-3}$. The relations (7), (10) and (11) solve our problem. We have found that the local evolution equations (1) admitting the Miura transformation (2) constitute the continual class

$$u_t = (D_x^3 + 2uD_x + u_1)p(x, t, u, u_1, \dots, u_{n-3}), \quad (12)$$

the corresponding local evolution equation (3) being

$$v_t = (D_x^2 + vD_x + v_1)p(x, t, v_1 - \frac{1}{2}v^2, D_x(v_1 - \frac{1}{2}v^2), \dots, D_x^{n-3}(v_1 - \frac{1}{2}v^2)), \quad (13)$$

where the function p and the order n are arbitrary.

Let us remind that this is a result of our old paper [1].

Zero-curvature representations

The continual class (12) originates from a different problem as well. Let us find all the local evolution equations (1) which admit ZCRs

$$D_t X - D_x T + [X, T] = 0 \quad (14)$$

with the fixed matrix X ,

$$X = \begin{pmatrix} 0 & u \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad (15)$$

and any 2×2 traceless matrices $T(x, t, u, u_1, \dots, u_{n-1})$, where D_t and D_x stand for the total derivatives, the square brackets denote the matrix commutator, and $u_i = \partial_x^i u$ ($i = 1, 2, \dots$). We solve this problem by the cyclic basis method [3, 4, 5].

In the present case of the matrix X given by (15), the characteristic form of the ZCR (14) of an evolution equation (1) is

$$fC = \nabla T, \quad (16)$$

where f is the right-hand side of (1), $C = \partial X / \partial u$, and the operator ∇ is defined as $\nabla M = D_x M - [X, M]$ for any 2×2 matrix M . Computing C , ∇C , $\nabla^2 C$ and $\nabla^3 C$, we find that the cyclic basis is $\{C, \nabla C, \nabla^2 C\}$, with the closure equation

$$\nabla^3 C = -u_1 C - 2u \nabla C. \quad (17)$$

Decomposing T over the cyclic basis as

$$T = a_0 C + a_1 \nabla C + a_2 \nabla^2 C, \quad (18)$$

where a_0 , a_1 and a_2 are functions of $x, t, u, u_1, \dots, u_{n-1}$, we find from (16) and (17) that

$$a_1 = -D_x a_2, \quad a_0 = -D_x a_1 + 2u a_2, \quad f = D_x a_0 - u_1 a_2. \quad (19)$$

Note that the function $a_2 = p(x, t, u, u_1, \dots, u_{n-3})$ and the order n remain undetermined. The explicit expressions

$$T = \begin{pmatrix} \frac{1}{2} D_x p & D_x^2 p + u p \\ -\frac{1}{2} p & -\frac{1}{2} D_x p \end{pmatrix} \quad (20)$$

and

$$f = D_x^3 p + 2u D_x p + u_1 p, \quad (21)$$

which follow from (18) and (19), solve our problem.

We have found that the local evolution equations (1) admitting ZCRs (14) with the matrix X given by (15) constitute the continual class (12), the corresponding matrices T being determined by (20).

References

- [1] S. Yu. Sakovich. The Miura transformation and Lie-Bäcklund algebras of exactly solvable equations. Phys. Lett. A 132(1):9–12 (1988).

- [2] R. M. Miura. Korteweg–de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *J. Math. Phys.* 9(8):1202–1204 (1968).
- [3] S. Yu. Sakovich. On zero-curvature representations of evolution equations. *J. Phys. A: Math. Gen.* 28(10):2861–2869 (1995).
- [4] S. Yu. Sakovich. True and fake Lax pairs: how to distinguish them. *ArXiv nlin.SI/0112027* (2001).
- [5] S. Yu. Sakovich. Cyclic bases of zero-curvature representations: five illustrations to one concept. *Acta Appl. Math.* 83(1–2):69–83 (2004); *arXiv nlin.SI/0212019* (2002).
- [6] S. A. Igonin. Miura type transformations and homogeneous spaces. *ArXiv nlin.SI/0412036* (2004).